# Math 261 B Tues. Dec. 1

$\mathcal{U}_q(\mathfrak{g})$ $\qquad$ $\mathcal{A} = \mathbb{Q}(q)$

$X \ni \alpha_i, \quad \alpha_i^\vee \in X^*$ $\qquad$ Cartan data for $G$

Integers $d_i > 0$ s.t. $\quad d_i \langle \alpha_i^\vee, \alpha_j \rangle = d_j \langle \alpha_j^\vee, \alpha_i \rangle$

Generators : $\underset{\mathcal{O}_{\mathcal{A}}(T^\vee)}{\underset{\cap}{K^\beta}}$ $(\beta \in X^*)$ , $E_i$, $F_i$ $\quad i = 1, \dots, n$

Relations :

$$K^\beta E_i K^{-\beta} = q^{\langle \beta, \alpha_i \rangle} E_i, \quad K^\beta F_i K^{-\beta} = q^{\langle \beta, -\alpha_i \rangle} F_i$$

$$[E_i, F_j] = 0 \qquad i \neq j$$

$$[E_i, F_i] = \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} \qquad q_i = q^{d_i} \quad K_i = K^{d_i \alpha_i^\vee}$$

$(E_i, F_i, K_i^{\pm 1})$ generate a $\mathcal{U}_{q_i}(\mathfrak{sl}_2)$

Quantum Serre Relations

# Classical Serre relations

$i \neq j$ In $\mathfrak{g}$, $[F_i, E_j] = 0$ b/c $\alpha_j - \alpha_i$ isn't a root

$\Rightarrow$ For $\mathrm{Ad} : \mathcal{U}(\mathfrak{sl}_2)_i \curvearrowright \mathfrak{g}$

$(\mathrm{Ad}\, F_i)\, E_j = 0$

$\Rightarrow$ $E_j$ is a lowest wt. vector of weight $\langle \alpha_i^\vee, \alpha_j \rangle = -m$

$\Rightarrow$ $E_j$ sits in a copy of $V^m \subset \mathfrak{g}$ for $(\mathfrak{sl}_2)_i$

$$(\mathrm{Ad}\, E_i)^{m+1} E_j = 0$$

$$\| \quad (\mathrm{ad}\, E_i)^{m+1} E_j$$

$$\| \quad \sum_{k+l=m+1} (-1)^l \binom{m+1}{k} E_i^k E_j E_i^l$$

$\mathrm{ad}(x)\, y = [x,y]$

$\mathcal{U}(\mathfrak{g})$ $\Delta x = x \otimes 1 + 1 \otimes x$

$Sx = -x$

$\mathrm{Ad} : \mathcal{U}(\mathfrak{g}) \curvearrowright \mathcal{U}(\mathfrak{g})$

$(\mathrm{Ad}\, x)\, y = \sum x_{(1)}\, y\, S x_{(2)}$

$\Delta x = \sum x_{(1)} \otimes x_{(2)}$

$x$ primitive $\Rightarrow$

$(\mathrm{Ad}\, x)\, y = xy - yx$

$= \mathrm{ad}(x)\, y$

## Quantum version

Compute $S(F_i)$

$$\mu(1\otimes S)\Delta F_i = \varepsilon(F_i) = 0$$

$$\| $$

$$\mu(1\otimes S)(F_i \otimes K_i^{-1} + 1\otimes F_i)$$

$S(K^\beta) = K^{-\beta}$

$\Delta K^\beta = K^\beta \otimes K^\beta$

$$F_i K_i + SF_i = 0$$

$$\Rightarrow S(F_i) = -F_i K_i$$

$$\Rightarrow (\mathrm{Ad}\, F_i)x = F_i x K_i - x F_i K_i = [F_i, x] K_i$$

$[F_i, E_j] = 0 \Rightarrow (\mathrm{Ad}\, F_i) E_j = 0$

$E_j$ is also a $K_i$ weight vector of weight $\alpha_j$

Want $E_j$ to be bottom of a $U_{q_i}(sl_2)_i$ string $\cong V^m$

$$K_i E_j K_i^{-1} = q^{d_i \langle \alpha_i^\vee, \alpha_j \rangle} E_j$$

$$= q^{-\dim} = q_i^{-m}$$

Then $(\mathrm{Ad}\, E_i)^{m+1} E_j = 0$ for $i \neq j$

i.e. weight $-m$ for $U_{q_i}(sl_2)_i$

← Quantum Serre Rels in $E_1, \dots, E_n$

Symmetry of the Hopf algebra:

$K^\beta$ fixed, $\quad E_i \longrightarrow F_i K_i$

$F_i \longrightarrow -K_i^{-1} E_i$

Can check: preserves $\Delta$, and all other relations

$\longrightarrow$ Quantum serre in $\mathcal{F}$ : $\quad (\mathrm{Ad}\, F_i K_i)^{m+1} (F_j K_j) = 0$

$\mathrm{Ad}\, F_i K_i = (\mathrm{Ad}\, F_i)(\mathrm{Ad}\, K_i)$

$\longleftarrow$ any weight vector is an eigenvector

Equivalently $\quad (\mathrm{Ad}\, F_i)^{m+1} (F_j K_j) = 0$

Ex. If $m=0$ $\quad \langle \alpha_i^\vee, \alpha_j \rangle = 0$ : classically, $E_i, E_j$ commute

Quantum: $\quad (\mathrm{Ad}\, E_i) E_j = 0 = E_i E_j - q_i^{\overset{\langle \alpha_i^\vee, \alpha_j \rangle}{\underset{0}{\|}}} E_j E_i = [E_i, E_j]$

$$\left[ \begin{array}{l} \Delta E_i = E_i \otimes 1 + K_i \otimes E_i \\ \Rightarrow S(E_i) = -K_i^{-1} E_i \end{array} \right]$$

$q_i^{\langle \alpha_i^\vee, \lambda \rangle} x$ if $x$ has weight $\lambda$

$\Rightarrow (\mathrm{Ad}\, E_i)\, x = E_i x - \underbrace{K_i x K_i^{-1}} E_i = K_i [K_i^{-1} E, x]$

If $m=1$ $\quad E_j \xrightarrow[\mathrm{Ad}\,E_i]{} E_iE_j - q_i^{-1}E_jE_i \xrightarrow[\mathrm{Ad}\,E_i]{} E_i(E_iE_j - q_i^{-1}E_jE_i) - q_i(E_iE_j - q_i^{-1}E_jE_i)E_i$

$K_iE_jK_i^{-1} = q_i^{-1}E_j$

$K_iE_iK_i^{-1} = q_i^2E_i$

$= E_i^2E_j - (2)_{q_i}E_iE_jE_i + E_jE_i^2$

$(2)_{q_i} = q_i + q_i^{-1}$ $\quad$ (classical : $E_i^2E_j - 2E_iE_jE_i + E_jE_i^2$)

Exercise $\quad \left(\mathrm{Ad}\,E_i^{(m+1)}\right)E_j = \sum_{k+l=m+1} (-1)^l E_i^{(k)}E_jE_i^{(l)}$

$E_i^{m+1}/(m+1)_{q_i}!$

$\mathbb{N}^n$

Why? $\quad$ Answer #1 $\quad K^\beta \quad \beta \in X^*$

$\mathcal{U}_+ = \mathcal{O}_q(T^\vee) \otimes \mathcal{A}\langle E_1, \ldots, E_n\rangle$ $\qquad K^\beta E_iK^{-\beta} = q^{\langle\beta,\alpha_i\rangle}E_i$

$\mathcal{U}_- = \mathcal{O}_q(T) \otimes \mathcal{A}\langle F_1, \ldots, F_n\rangle$ $\qquad K^\beta E_i = q^{\langle\beta,\alpha_i\rangle}E_iK^\beta$

$x^\lambda \quad \lambda \in X$

$x^\lambda F_i x^{-\lambda} = q^{(\lambda,\alpha_i)}F_i$

where $(\lambda,\alpha_i) \underset{\mathrm{def}}{=} d_i\langle\alpha_i^\vee, \lambda\rangle$

Have dual pairing as Hopf algebras :

Usual $\langle K^\beta, x^\lambda \rangle = q^{\langle \beta, \lambda \rangle}$ $\qquad$ $(\alpha_j, \alpha_i) = (\alpha_i, \alpha_j)$

on $\mathcal{O}_q(T^\vee)$, $\mathcal{O}_q(T)$, $\qquad$ $X \otimes Q \to \mathbb{Z}$ s.t. $Q \otimes Q \to \mathbb{Z}$ is symmetric.

$\langle E_i, x^\lambda \rangle = 0$

$\langle E_i, F_j \rangle = \delta_{ij}$

$\langle K^\beta, F_i \rangle = 0$

$\mathcal{J}_+ \longrightarrow (\mathcal{J}_-)^*$ $\qquad$ Kernel is the 2-sided ideal generated by Q-Serre relations.

Answer #2 $\quad$ Integrability $\quad$ suppose $\mathcal{U}_q(\mathfrak{g}) \curvearrowright V$

$v \in V$ $\quad$ contained in a finite dim'l, standard module

(weight vector) $\quad$ for each $\mathcal{U}_{q_i}(\mathfrak{sl}_2)_i$

$K^\beta v = q^{\langle \beta, \lambda \rangle} v$ $\quad$ i.e. killed by some power of $E_i$, $F_i$

$\forall \beta$

Want $E_j v$ to again be integrable:

$F_j$ is OK by $(\mathfrak{sl}_2)_j$ $\qquad$ $F_{i \neq j}$ are OK b/c commute with $E_j$

$E_j$ is OK. Problem is the other $E_i$'s. $\mathbb{Q}$-Serre relations fix it:

$$E_i^M \; E_j v \qquad\qquad E_i^N v = 0 \qquad\qquad x\,y\,v \quad x,y \in \mathcal{U}$$

$$= \Sigma_{(}\left(\mathrm{Ad}\left(E_i^M\right)_{(1)} E_j\right) \cdot \mathrm{Ad}\left(E_i^M{}_{(2)}\right) v \qquad\qquad = \Sigma\left((\mathrm{Ad}\,x_{(1)})\,y\right)\,x_{(2)} v$$

$$\Delta E_i^M = \left(\Delta E_i\right)^M = \left(E_i \otimes 1 + K_i \otimes E_i\right)^M$$

$= \text{sum of terms of the form } q^{?} E_i^k K_i^{\ell} \otimes E_i^{\ell}$

$k + \ell = M$

$$q^{?} \qquad E_i^M{}_{(1)} = E_i^k K_i^{?} \qquad\qquad E_i^M{}_{(2)}\; E_i^{\ell}$$

All terms with $k > m$ or $\ell \geq N$ vanish

If $M \geq m + N$, it all vanishes: $E_i^M E_j v = 0$

# Representations of $U_q(\mathfrak{g})$

Ex. $V^m$ for $U_q(sl_2)$


Ex. For $U_q(sl_3)$